Dynamika v MKP

Systém pružina-hmota pod účinkom zaťaženia v čase.

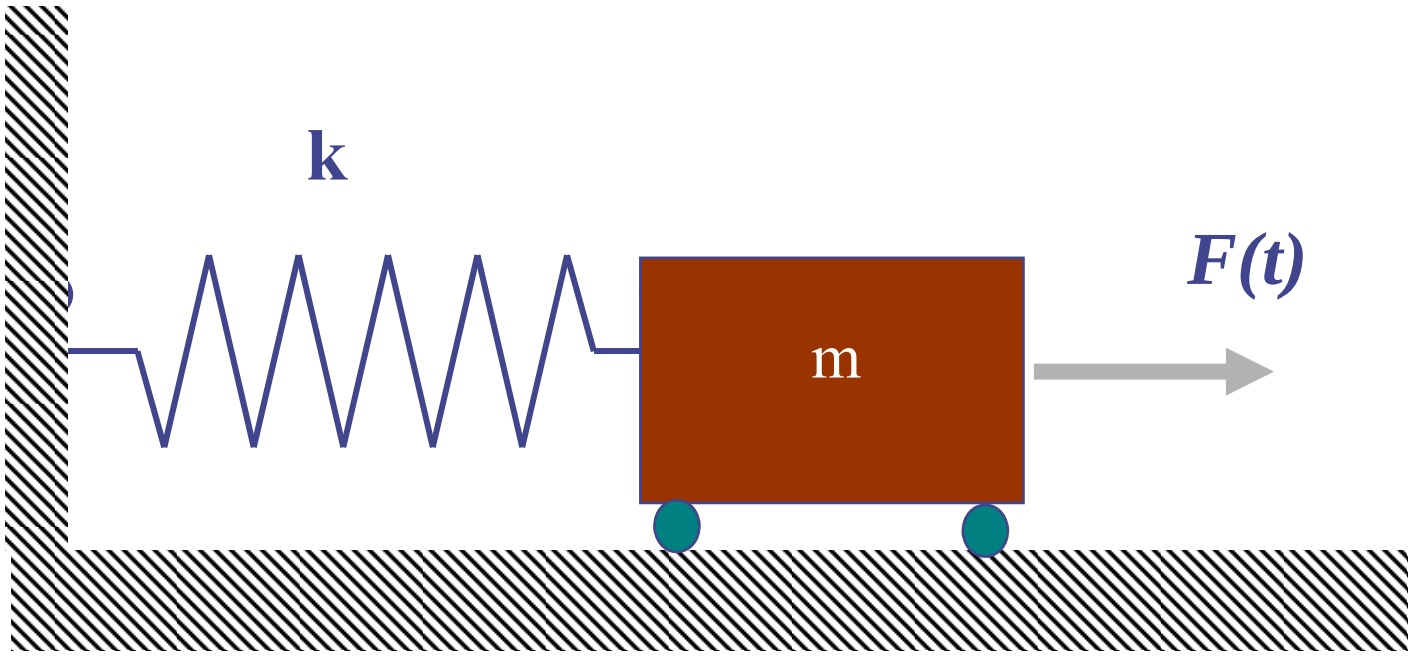
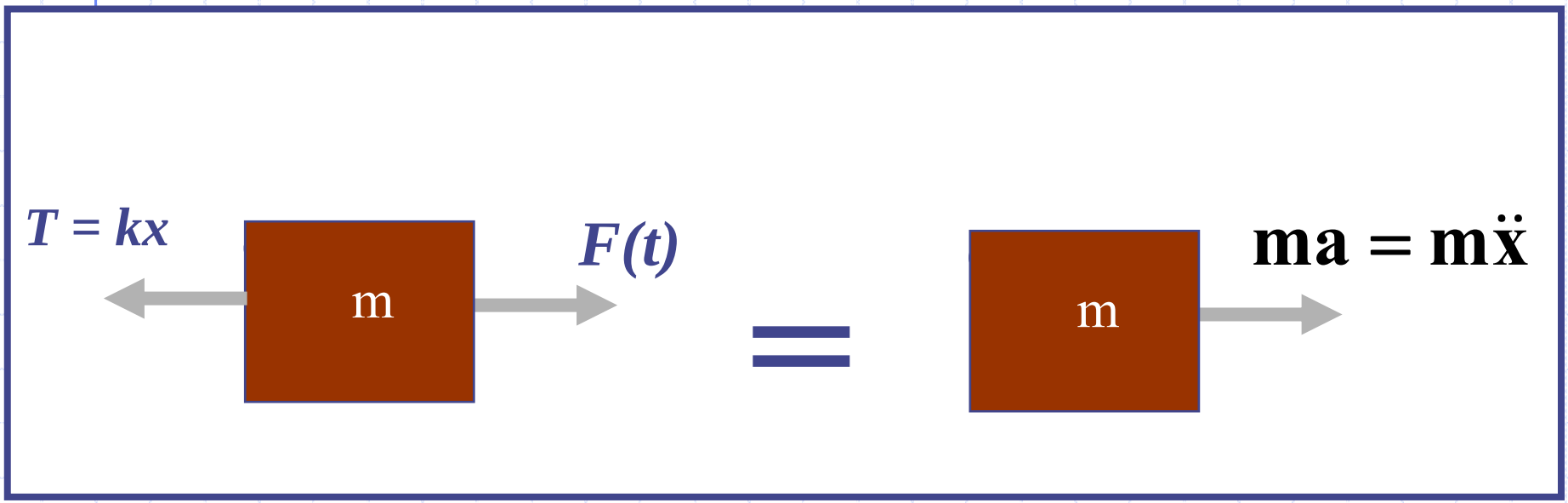


Schéma voľného hmotného telesa



$$F(t) - k\,x = m\,\ddot{x}$$

$$m\,\ddot{x} + k\,x = F(t)$$

Riešenie dif.rovnice je rovné súčtu homogénneho a partikulárneho riešenia:

Homogénne:

$$F(t) = 0$$

$$m\ddot{x} + k\,x = 0$$

$$Nech:$$

$$\omega^2 = \frac{k}{m}$$

$$Potom:$$

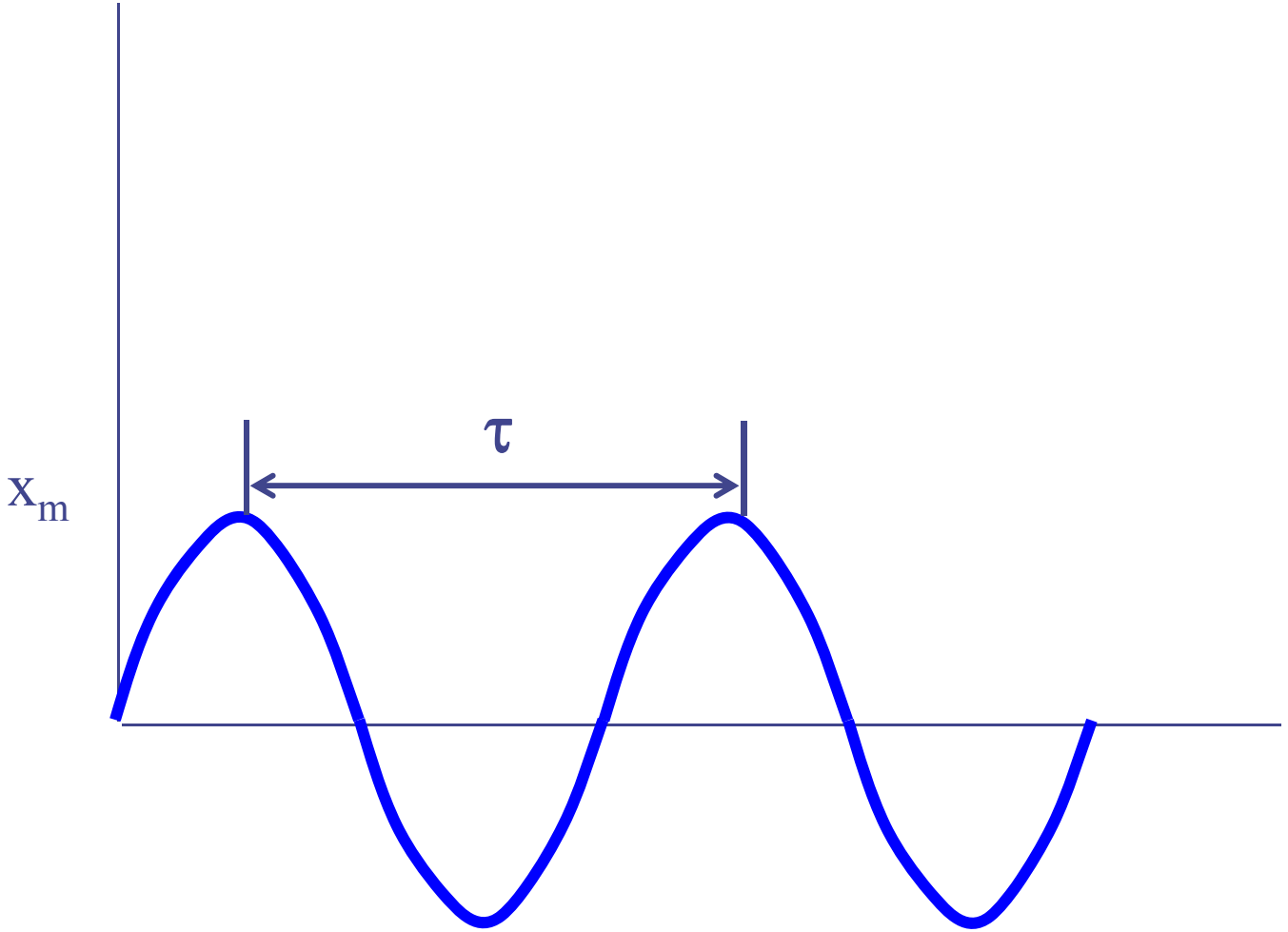
$$\ddot{x} + \omega^2 x = 0$$

ω^2 je prirodzená kruhová frekvencia

$$\tau = \frac{2\pi}{\omega}$$

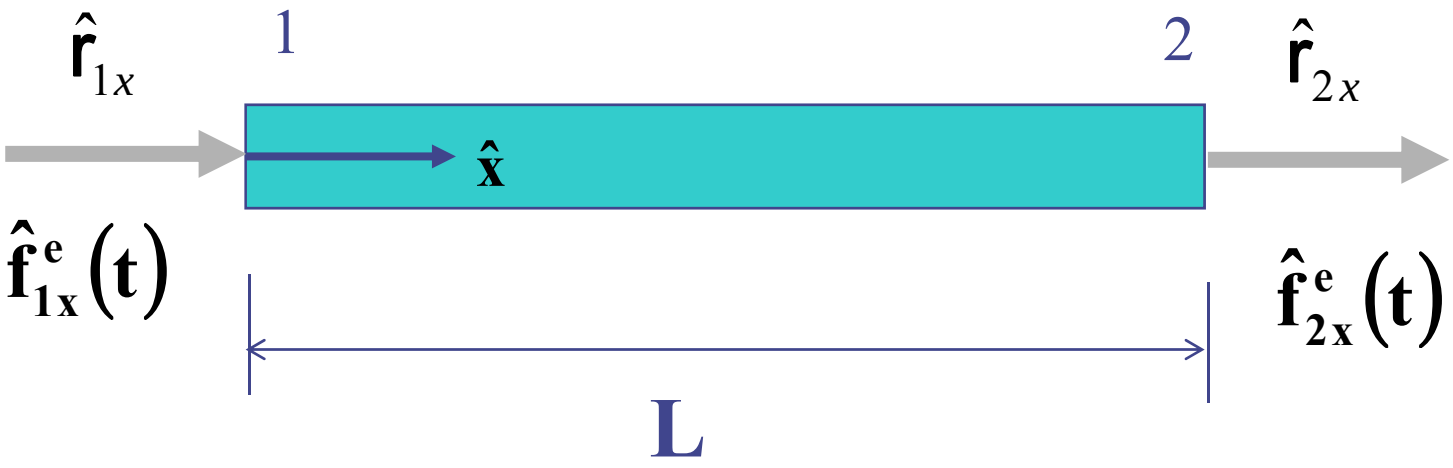
τ je perióda [s]

Priebeh posunu pri harmonickom pohybe



Jednodimenziálny - prútový prvok

Step 1 - Select Element Type



E - modulus of elasticity
A - cross-sectional area
ρ - mass density

Step 2 - Select a Displacement Function

$$\hat{u} = a_1 + a_2 \hat{x}$$

$$\hat{u} = N_1 \hat{r}_{1x} + N_2 \hat{r}_{2x}$$

$$N_1 = 1 - \frac{\hat{x}}{L}$$

$$N_2 = \frac{\hat{x}}{L}$$

Step 3 - Define Strain/Displacement and Stress/Strain Relationships

$$\{\varepsilon_x\} = \frac{\partial \hat{u}}{\partial \hat{x}} = [B]\{\hat{r}\}$$

$$[B] = \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix}$$

$$\{\hat{r}\} = \begin{Bmatrix} \hat{r}_{1x} \\ \hat{r}_{2x} \end{Bmatrix}$$

$$\{\sigma\} = [D]\{\varepsilon_x\} = [D][B]\{\hat{r}\}$$

Step 4 - Derive Element Stiffness and Mass Matrices and Equations

With time dependent loading

$$\hat{\mathbf{f}}_{1x} \neq \hat{\mathbf{f}}_{2x}$$

Newton's Second Law

$$\vec{f} = m\vec{a}$$

NODAL EQUILIBRIUM EQUATIONS

$$\hat{f}_{1x}^{e} = \hat{f}_{1x} + m_1 \frac{\partial^2 \hat{r}_{1x}}{\partial t^2}$$

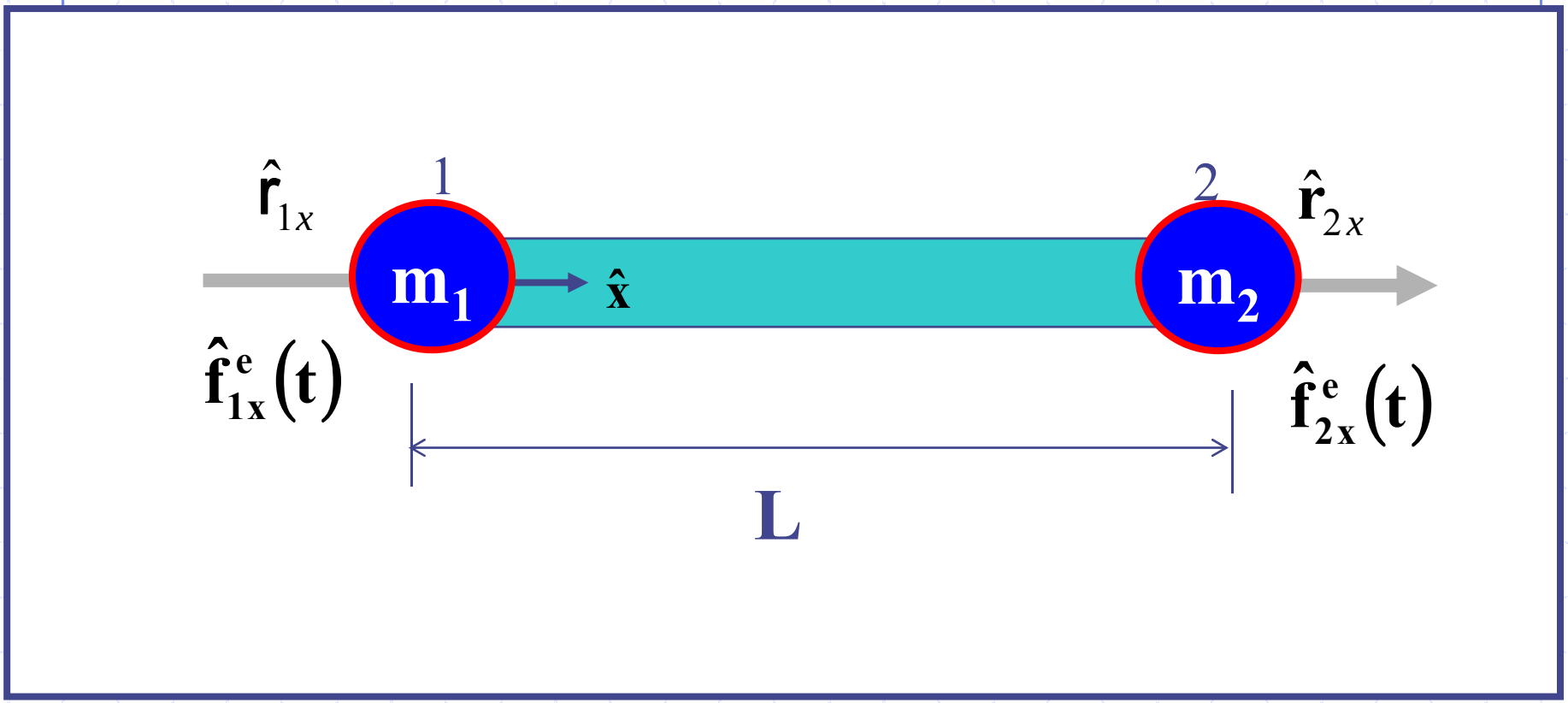
$$\hat{f}_{2x}^{e} = \hat{f}_{2x} + m_2 \frac{\partial^2 \hat{r}_{2x}}{\partial t^2}$$

m_1 and m_2 are obtained by lumping the total mass of the bar equally at the two nodes

$$m_1 = \frac{\rho AL}{2}$$

$$m_2 = \frac{\rho AL}{2}$$

Lumped Mass Model



Equilibrium in Matrix Form

$$\begin{Bmatrix} \hat{f}^{e}_{1x} \\ \hat{f}^{e}_{2x} \end{Bmatrix} = \begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{2x} \end{Bmatrix} + \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \dfrac{\partial \hat{r}_{1x}}{\partial t^2} \\ \dfrac{\partial \hat{r}_{2x}}{\partial t^2} \end{Bmatrix}$$

Equilibrium in Matrix Form

$$\{\hat{f}^{e}(t)\} = [\hat{k}]\{\hat{r}\} + [\hat{m}]\{\ddot{\hat{r}}\}$$

Defining Terms

$$[k] = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$ Element Stiffness Matrix

$$[m] = \frac{\rho AL}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$ Element Lumped Mass Matrix

$$\{\hat{r}\}$$ Nodal Displacements

$$\{\ddot{\hat{r}}\} = \frac{\partial^2 \{\hat{r}\}}{\partial t^2}$$ Nodal Accelerations

Consistent Mass Matrix

$$\{\mathbf{X}^{\mathbf{e}}\} = -\rho\{\ddot{\hat{\mathbf{u}}}\}$$

$$\{\mathbf{f_b}\} = \iiint_{\mathbf{V}} [\mathbf{N}]^{\mathbf{T}} \{\mathbf{X}\}\,\mathbf{dV}$$

$$\{\mathbf{f_b}\} = -\iiint_{\mathbf{V}} \rho[\mathbf{N}]^{\mathbf{T}} \{\ddot{\hat{\mathbf{u}}}\}\mathbf{dV}$$

Consistent Mass Matrix

$$\{\hat{u}\} = [N]\{\hat{r}\}$$

$$\{\dot{\hat{u}}\} = [N]\{\dot{\hat{r}}\}$$

$$\{\ddot{\hat{u}}\} = [N]\{\ddot{\hat{r}}\}$$

Consistent Mass Matrix

$$\{f_b\} = -\iiint_V \rho [N]^T [N] \{\ddot{\hat{r}}\} dV$$

$$\{f_b\} = -[\hat{m}] \{\ddot{\hat{r}}\}$$

$$[\hat{m}] = \iiint_V \rho [N]^T [N] dV$$

Consistent Mass Matrix
Bar Element

$$[\hat{\mathbf{m}}] = \iiint_V \rho [\mathbf{N}]^T [\mathbf{N}] \, dV$$

$$[\hat{\mathbf{m}}] = \iiint_V \rho \begin{Bmatrix} 1 - \frac{\hat{x}}{L} \\ \frac{\hat{x}}{L} \end{Bmatrix} \begin{bmatrix} 1 - \frac{\hat{x}}{L} & \frac{\hat{x}}{L} \end{bmatrix} dV$$

$$[\hat{\mathbf{m}}] = \rho A \int_0^L \begin{Bmatrix} 1 - \frac{\hat{x}}{L} \\ \frac{\hat{x}}{L} \end{Bmatrix} \begin{bmatrix} 1 - \frac{\hat{x}}{L} & \frac{\hat{x}}{L} \end{bmatrix} d\hat{x}$$

Consistent Mass Matrix
Bar Element

$$[\hat{\mathbf{m}}] = \rho \mathbf{A} \int_0^{\mathbf{L}} \begin{bmatrix} \left(1-\frac{\hat{\mathbf{x}}}{\mathbf{L}}\right)\left(1-\frac{\hat{\mathbf{x}}}{\mathbf{L}}\right) & \left(1-\frac{\hat{\mathbf{x}}}{\mathbf{L}}\right)\left(\frac{\hat{\mathbf{x}}}{\mathbf{L}}\right) \\ \left(1-\frac{\hat{\mathbf{x}}}{\mathbf{L}}\right)\left(\frac{\hat{\mathbf{x}}}{\mathbf{L}}\right) & \left(\frac{\hat{\mathbf{x}}}{\mathbf{L}}\right)\left(\frac{\hat{\mathbf{x}}}{\mathbf{L}}\right) \end{bmatrix} \mathbf{d}\hat{\mathbf{x}}$$

$$[\hat{\mathbf{m}}] = \frac{\rho \mathbf{A} \mathbf{L}}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

STEP 5 - Assemble the Global Equations and Apply B.C.'s

$$\{F(t)\} = [K]\{r\} + [M]\{\ddot{r}\}$$

$$\{F(t)\} = [K]\{d\} + [M]\{\ddot{d}\}$$

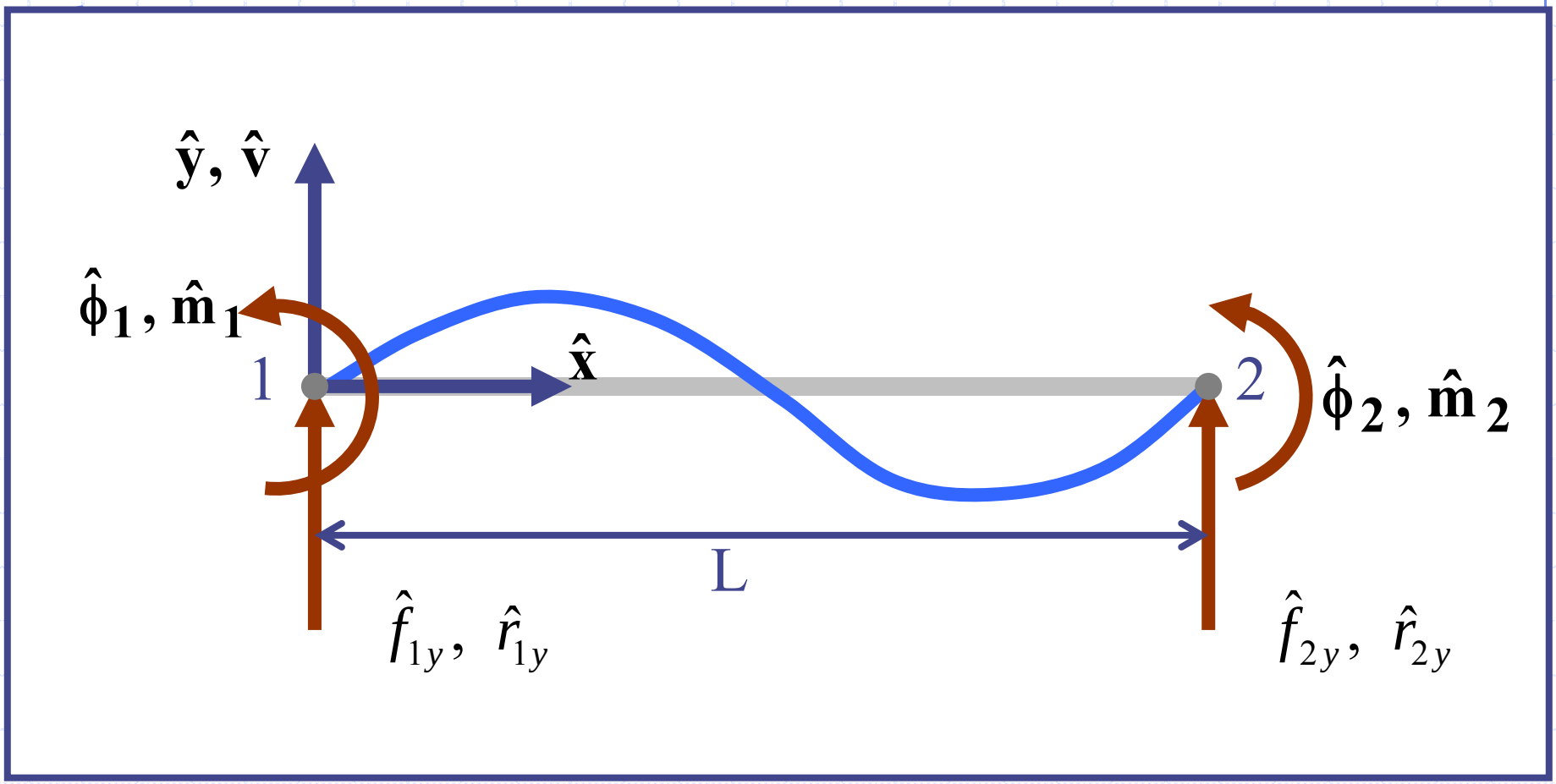
Now must solve coupled set of ODE's instead of set of linear algebraic equations!

Consistent Mass Matrix

$$[\mathbf{m}] = \iiint_{V} \rho [\mathbf{N}]^{T} [\mathbf{N}] \, \mathbf{dV}$$

One Dimensional Beam Element

Beam Element



Shape Functions

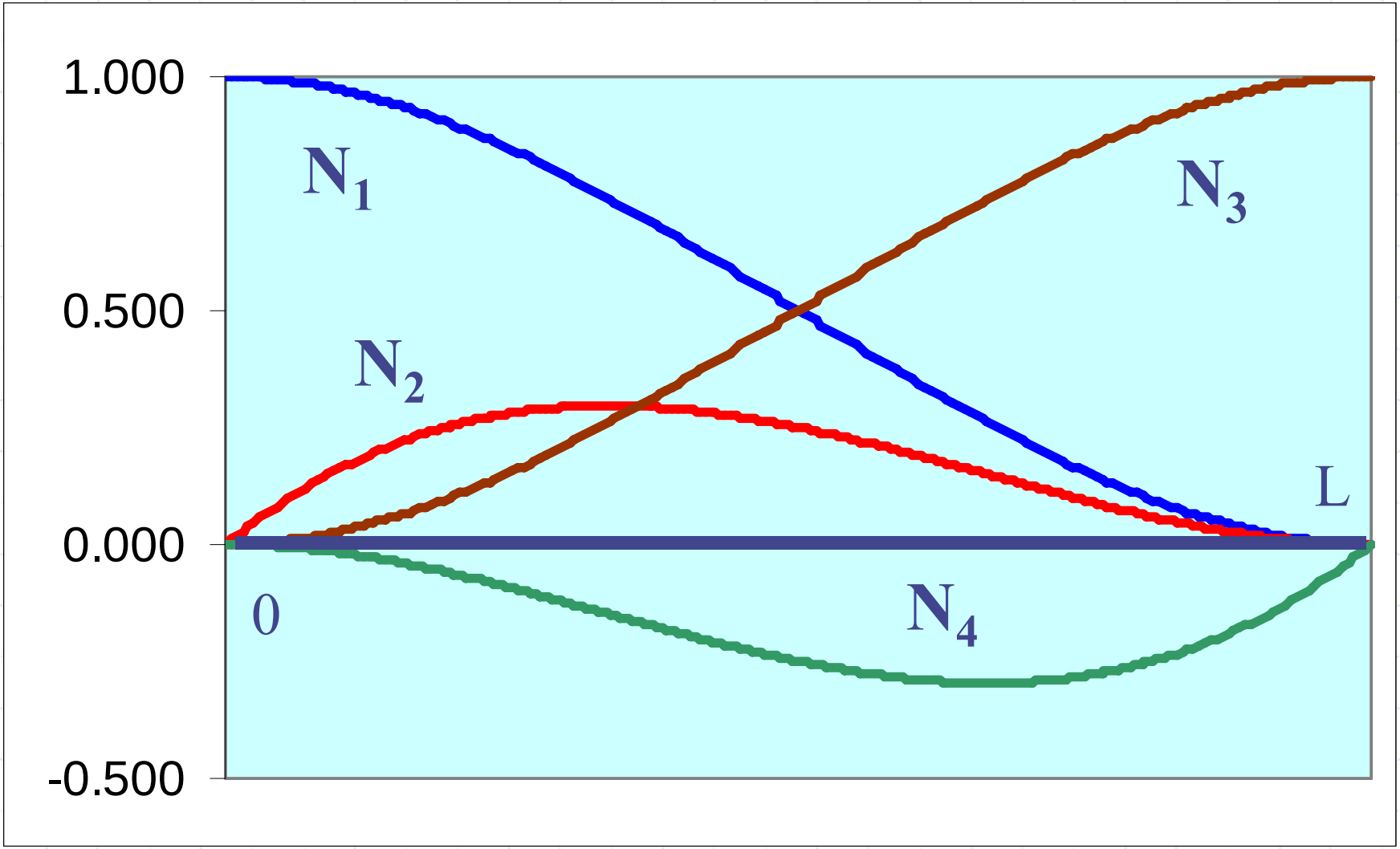
$$N_1 = \frac{1}{L^3}\left(2\hat{x}^3 - 3\hat{x}^2 L + L^3\right)$$

$$N_2 = \frac{1}{L^3}\left(\hat{x}^3 L - 2\hat{x}^2 L^2 + \hat{x} L^3\right)$$

$$N_3 = \frac{1}{L^3}\left(-2\hat{x}^3 + 3\hat{x}^2 L\right)$$

$$N_4 = \frac{1}{L^3}\left(\hat{x}^3 L - \hat{x}^2 L^2\right)$$

Shape Functions



Consistent Mass Matrix

$$[\mathbf{m}] = \iiint_V \rho [\mathbf{N}]^T [\mathbf{N}] \, dV$$

$$[\mathbf{m}] = \frac{\mathbf{m}}{420} \begin{bmatrix} 156 & 22\mathbf{L} & 54 & -13\mathbf{L} \\ 22\mathbf{L} & 4\mathbf{L}^2 & 13\mathbf{L} & -3\mathbf{L}^2 \\ 54 & 13\mathbf{L} & 156 & -22\mathbf{L} \\ -13\mathbf{L} & -3\mathbf{L}^2 & -22\mathbf{L} & 4\mathbf{L}^2 \end{bmatrix}$$

Lumped Mass Matrix

$$[m] = \frac{m}{2}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\alpha L^2}{210} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{\alpha L^2}{210} \end{bmatrix}$$

Lumped Mass Matrix

2nd and 4th terms account for rotary inertia.
$\alpha = 0$ if this is ignored.
$\alpha = 17.5$ if mass moment of inertia of bar spinning about one end is selected

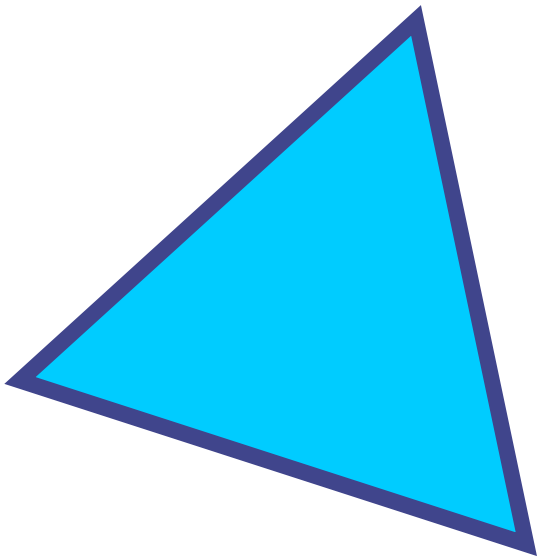
$$I = \frac{\left(\frac{m}{2}\right)\left(\frac{L}{2}\right)^2}{3}$$

Two Dimensional Plane Element

Consistent Mass Matrix - CST

$$[m] = \begin{bmatrix} Q & 0 \\ 0 & Q \end{bmatrix}$$

$$[Q] = \frac{m}{12} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{matrix} u_1 \\ u_2 \\ u_3 \end{matrix}$$

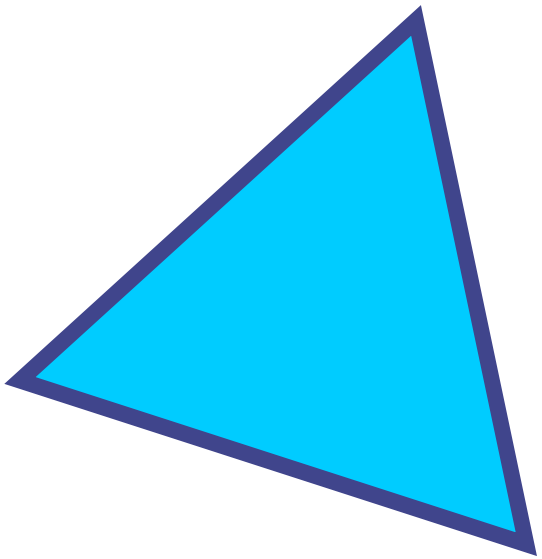


For each degree of freedom

Consistent Mass Matrix - CST

$$[m] = \frac{\rho A t}{12}\begin{bmatrix} 2 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 & 1 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 \\ 1 & 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 & 2 \end{bmatrix}$$

Lumped Mass Matrix - CST



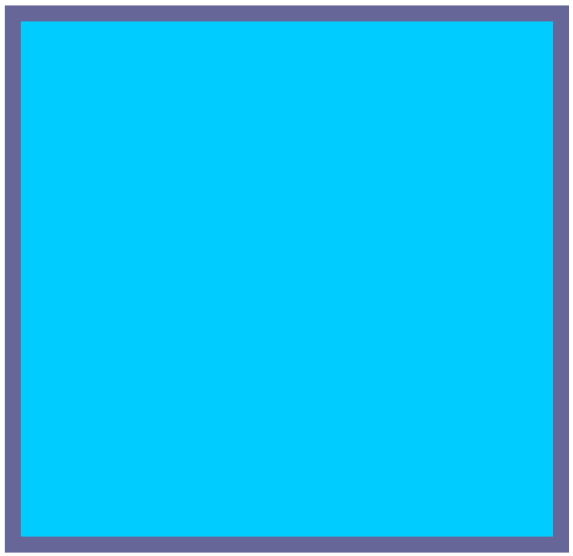
$$[m] = \left[\begin{array}{c|c} Q & 0 \\ \hline 0 & Q \end{array}\right]$$

$$[Q] = \frac{m}{3}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{matrix} u_1 \\ u_2 \\ u_3 \end{matrix}$$

Lumped Mass Matrix - CST

$$[\mathbf{m}] = \frac{\rho \mathbf{A} \mathbf{t}}{3} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Consistent Mass Matrix - Quad



$$[\mathbf{m}] = \begin{bmatrix} \mathbf{Q} & 0 \\ 0 & \mathbf{Q} \end{bmatrix}$$

$$[\mathbf{Q}] = \frac{\mathbf{m}}{36} \begin{bmatrix} 4 & 2 & 1 & 2 \\ 2 & 4 & 2 & 1 \\ 1 & 2 & 4 & 2 \\ 2 & 1 & 2 & 4 \end{bmatrix}$$

$$\mathbf{m} = \rho \mathbf{A} \mathbf{t}$$

Consistent Mass Matrix - Quad

$$[m] = \frac{m}{36}\begin{bmatrix} 4 & 0 & 2 & 0 & 1 & 0 & 2 & 0 \\ 0 & 4 & 0 & 2 & 0 & 1 & 0 & 2 \\ 2 & 0 & 4 & 0 & 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 4 & 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 & 4 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 & 0 & 4 & 0 & 2 \\ 2 & 0 & 1 & 0 & 2 & 0 & 4 & 0 \\ 0 & 2 & 0 & 1 & 0 & 2 & 0 & 4 \end{bmatrix}$$

Hybrid Methods

Hybrid Methods

Attempts have been made to combine consistent and lumped mass approaches to achieve some of the benefits of each!

HRZ Lumping

1. **H**inton, **R**ock, and **Z**ienkiewicz
2. Compute the diagonal terms of consistent mass matrix.
3. Compute total mass of element, ***m***
4. Compute ***s*** by adding diagonal coefficients associated with translational D-O-F that are in same direction.
5. Scale all diagonal coefficients by multiplying by ***m/s***

HRZ - Bar Element

$$[\hat{m}] = \frac{\rho A L}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$m = \rho A L$$

$$s = 4 \times \frac{\rho A L}{6}$$

$$\frac{m}{s} = \frac{3}{2}$$

$$[\hat{m}] = \frac{\rho A L}{6}\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

HRZ - Beam Element

$$[\mathbf{m}] = \frac{\mathbf{m}}{420}\begin{bmatrix} 156 & 22\mathbf{L} & 54 & -13\mathbf{L} \\ 22\mathbf{L} & 4\mathbf{L}^2 & 13\mathbf{L} & -3\mathbf{L}^2 \\ 54 & 13\mathbf{L} & 156 & -22\mathbf{L} \\ -13\mathbf{L} & -3\mathbf{L}^2 & -22\mathbf{L} & 4\mathbf{L}^2 \end{bmatrix}$$

$$\mathbf{m} = \rho\,\mathbf{A}\,\mathbf{L}$$

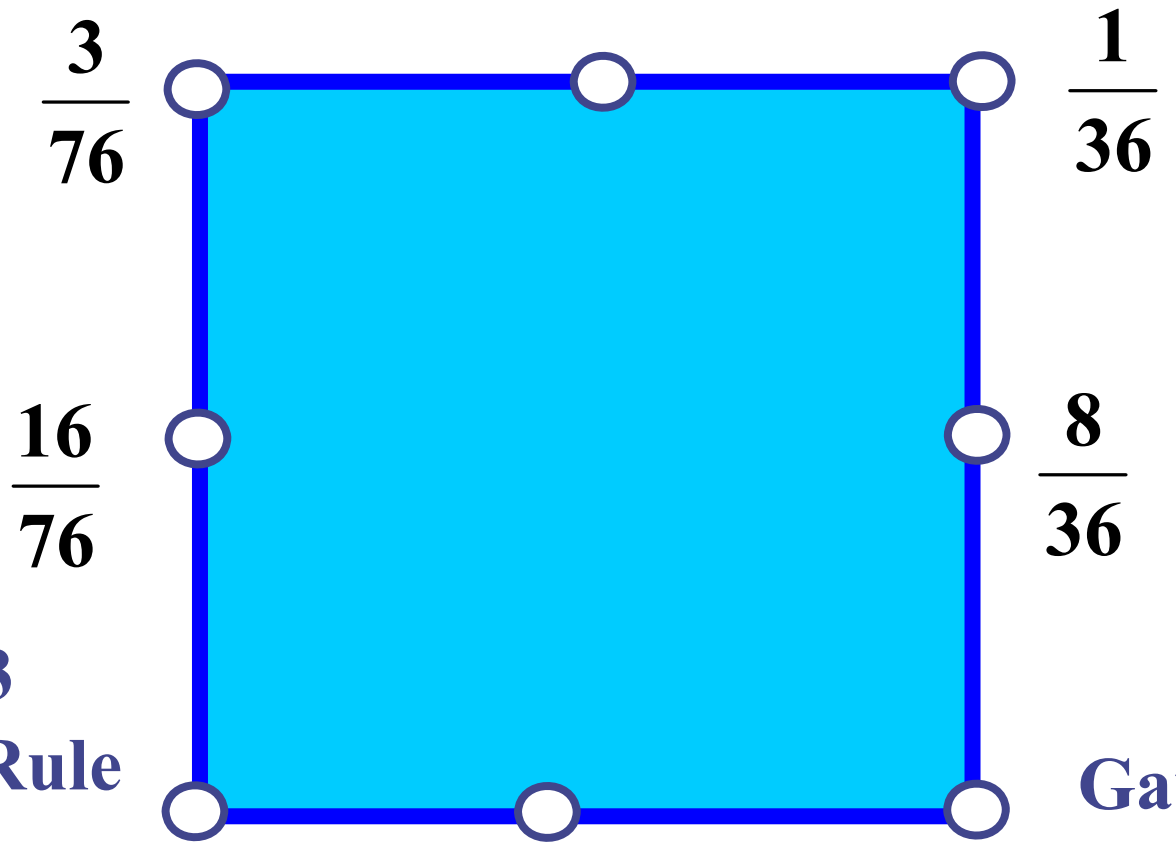
$$\mathbf{s} = 312 \times \frac{\rho\,\mathbf{A}\,\mathbf{L}}{420}$$

$$\frac{\mathbf{m}}{\mathbf{s}} = \frac{420}{312}$$

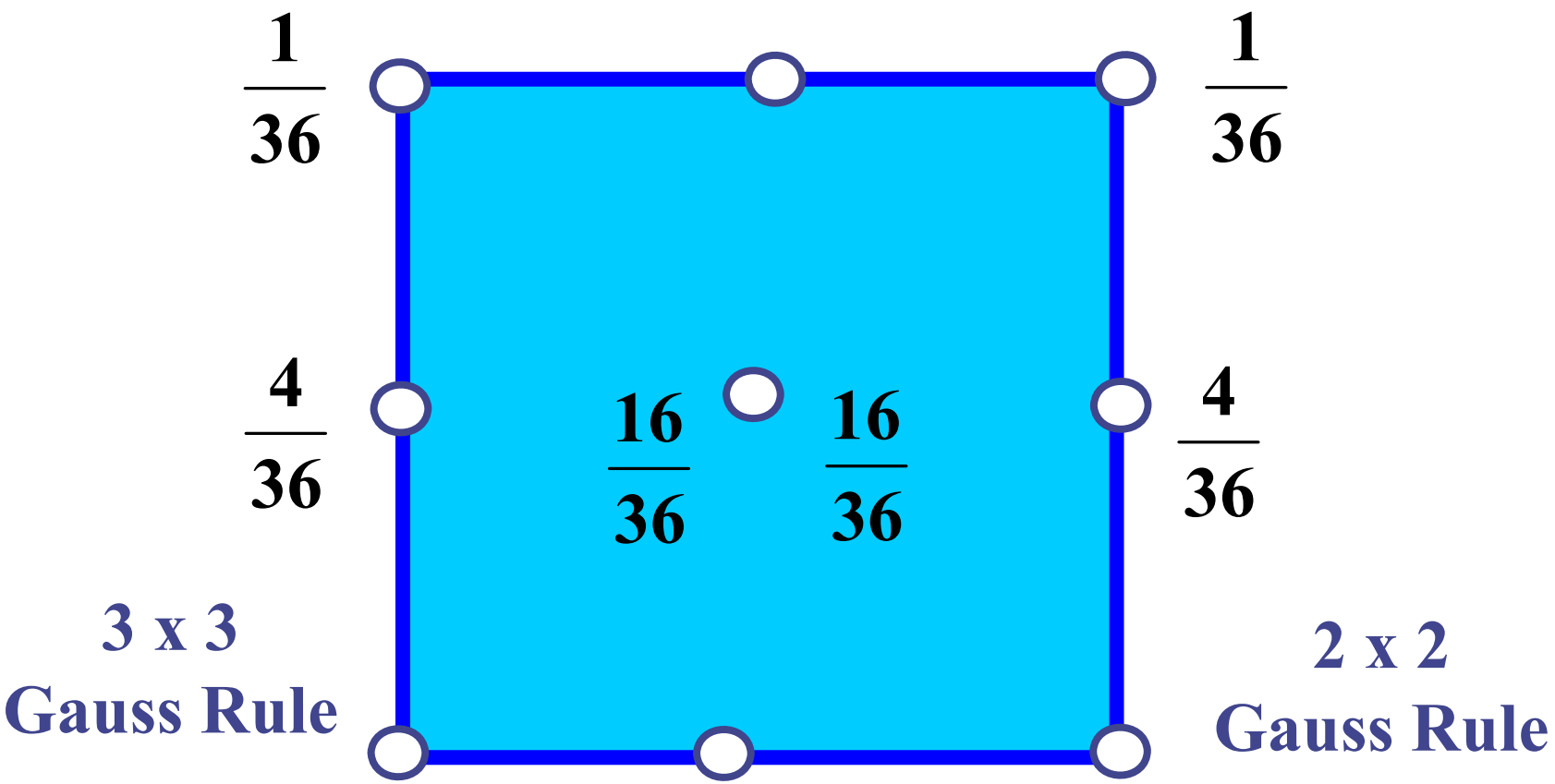
HRZ - Beam Element

$$[\hat{m}] = \frac{m}{420}\begin{bmatrix} \frac{420}{312}\times 156 & 0 & 0 & 0 \\ 0 & \frac{420}{312}\times 4L^2 & 0 & 0 \\ 0 & 0 & \frac{420}{312}\times 156 & 0 \\ 0 & 0 & 0 & \frac{420}{312}\times 4L^2 \end{bmatrix} = \frac{m}{78}\begin{bmatrix} 39 \\ L^2 \\ 39 \\ L^2 \end{bmatrix}$$

HRZ – Quadratic Serendipity



HRZ – Quadratic Lagrangian



% error in natural frequencies of a thick simply-supported plate.
Half of the plate modeled with 8-noded 24 d-o-f elements

Mode			Type of Mass Matrix		
m	n		Consistent (%)	HRz Lumping (%)	Ad Hoc Lumping (%)
1	1		0.11	0.32	0.32
2	1		0.4	0.45	0.45
2	2		0.35	2.75	4.12
3	1		5.18	0.05	5.75
3	2		4.68	2.96	10.15
3	3		13.78	5.18	19.42
4	2		16.88	1.53	31.7

Optimal Lumping

- Only translational d-o-f
- Based on consistent mass matrix
- Use appropriate quadrature rule
- Chose integration points to coincide with nodal locations
- [m] will be diagonal

- Let p be the highest order complete polynomial in shape function N
- let m be the highest order derivative in strain energy (m = 1 elasticity, m = 2 bending)
- Chose quadrature rule with degree of precision 2(p-m)

Three noded bar element

p = 2

m = 1

2(p-m) = 2

Three point quadrature rule.

Newton -Cotes has points at the nodes.

(Simpson' Rule)

$$\int_a^b f(x)dx = (b-a)\left[\frac{1}{6}f(a)+\frac{4}{6}f\left(\frac{b+a}{2}\right)+\frac{1}{6}f(b)\right]$$

$$m_{ij} = \rho A \int N_i N_j dx = \int_{-1}^{1} N_i N_j \, J \, d\xi$$

$$J = \frac{L}{2}$$

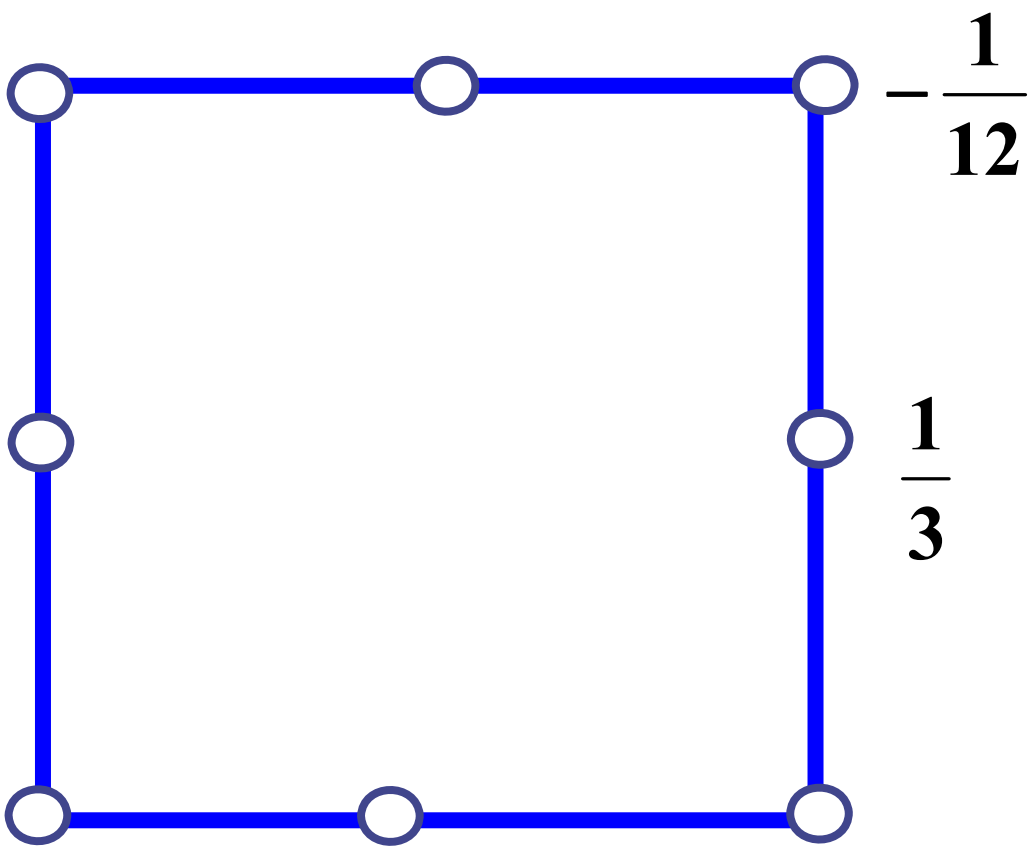
$$m_{ij} = \rho A \frac{L}{2}(1-(-1))\left[\frac{1}{6}N_i(-1)N_j(-1)+\frac{4}{6}N_i(0)N_j(0)+\frac{1}{6}N_i(1)N_j(1)\right]$$

$$i \neq j \quad m_{ij} = 0$$

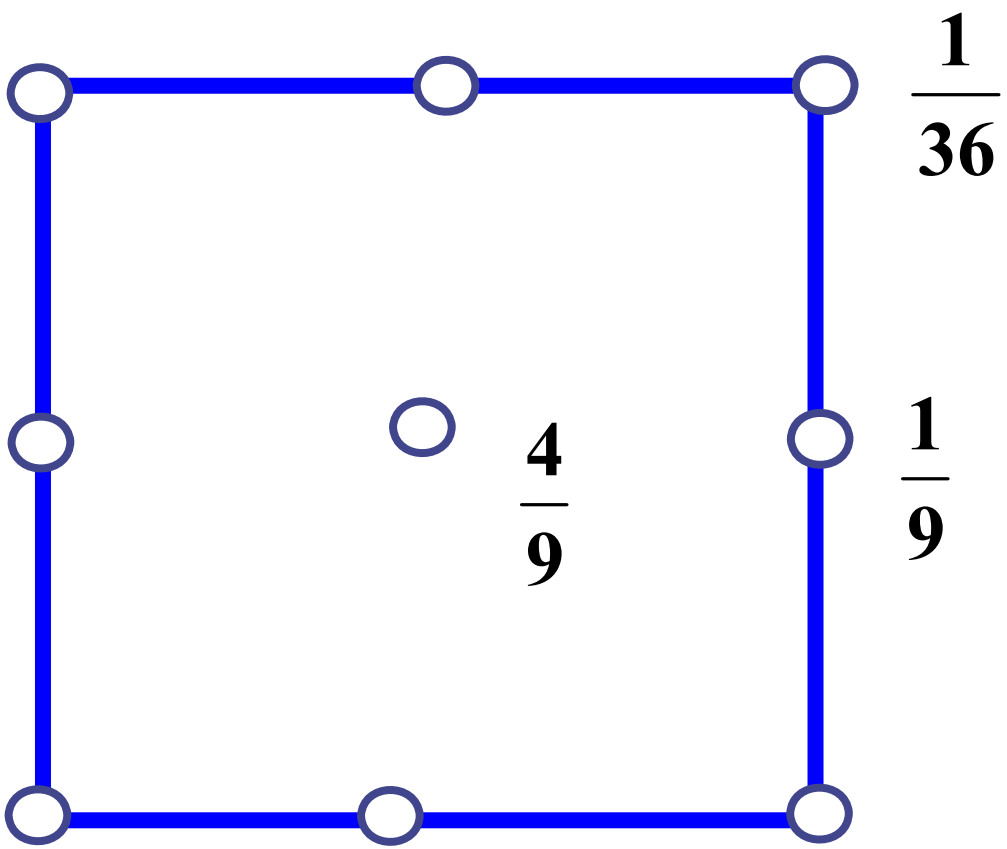
$$[m] = \frac{\rho AL}{6}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$



Serendipity



Lagrangian



Mass Matrices

- Product [m]{a} must yield the correct total force on an element (F = ma) when {a} represents a rigid-body translational acceleration.
- Consistent mass matrices, [m] and [M] are positive definite.
- Lumped mass matrix is positive semi-definite when zero terms appear on main diagonal.
- Lumped mass matrix is indefinite when negative terms appear on main diagonal.

Mass Matrices

- Special treatment may be needed to handle the last two cases.

Best Type ?

1. Consistent matrices usually more accurate for flexural problems.
2. Consistent matrices give upper bounds on natural frequencies.

Best Type ?

1. Lumped matrices usually give natural frequencies less than exact values.
2. Simpler to form.
3. Occupy less storage.
4. Require less computational effort.
5. Usually more important in time-history than in vibration problems.

Damping

Damping

1. Structural damping is not viscous.
2. Due to mechanisms such as hysteresis and slip in connections.
3. Mechanisms not well understood.
4. Awkward to incorporate into structural dynamic equations.
5. Makes equations computationally difficult.
6. Effects usually approximated by viscous damping.

Types of Damping Models

- Phenomenological Damping Methods (models actual dissipative mechanisms)
  - Elastic-Plastic Hysteresis Loss
  - Structural Joint Friction
  - Material Micro-cracking
- Spectral Damping Methods
  - Introduce Viscous Damping
  - Relies on Fraction of Critical Damping

Critical Damping

ξ *Fraction of Critical Damping*

$\xi = 1$ *Critical Damping*

Critical Damping marks the transition between oscillatory and non- oscillatory response of a structure

Critical Damping Ratio

$0.5\% \le \xi \le 5\%$	***Steel Piping***
$2\% \le \xi \le 15\%$	***Bolted or riveted steel structures***
$2\% \le \xi \le 15\%$	***Reinforced or Prestresses Concrete***

Actual value may depend on stress level.

Rayleigh or Proportional Damping

Damping matrix is a linear combination of stiffness and mass matrices:

$$[\mathbf{C}] = \alpha [\mathbf{K}] + \beta [\mathbf{M}]$$

Rayleigh or Proportional Damping

[C] is orthogonal damping matrix.
Modes may be uncoupled by eigenvectors associated with undamped problem.

$$\xi = \frac{1}{2}\left(\alpha\,\omega - \frac{\beta}{\omega}\right)$$

If critical damping ratio is known at two frequencies then:

$$\xi = \frac{1}{2}\left(\alpha\,\omega - \frac{\beta}{\omega}\right)$$

$$\alpha = 2\,\frac{\left(\xi_2\omega_2 - \xi_1\omega_1\right)}{\left(\omega_2^2 - \omega_1^2\right)}$$

$$\beta = 2\omega_1\omega_2\,\frac{\left(\xi_1\omega_2 - \xi_2\omega_1\right)}{\left(\omega_2^2 - \omega_1^2\right)}$$

Natural Frequencies and Mode Shapes Undamped, Unforced Response

$$\{D\} = \{\overline{D}\}\sin \omega t$$

$$\{\dot{D}\} = \omega\{\overline{D}\}\cos \omega t$$

$$\{\ddot{D}\} = -\omega^2 \{\overline{D}\}\sin \omega t$$

$\{\overline{D}\}$ *amplitudes of nodal d - o - f*

ω *circular frequency*

$$f = \frac{\omega}{2\pi} \quad (\text{Hz})$$

Eigenproblems

Results in generalized eigenproblem

$$([\mathbf{K}]-\lambda[\mathbf{M}])\{\overline{\mathbf{D}}\}=\{\mathbf{0}\}$$

$$\lambda=\omega^2$$

Trivial Solution:

$$\left| [K] - \lambda [M] \right| \neq 0$$

$$\{\overline{D}\} = 0$$

Nontrivial Solution:

$$\left| [\mathbf{K}] - \lambda [\mathbf{M}] \right| = 0$$

$$\{\overline{\mathbf{D}}\} \neq 0$$

$$\lambda_i \equiv \textit{Roots of Characteristic Polynomial (eigenvalues)}$$

$$\{\overline{\mathbf{D}}\}_i \equiv \textit{Associated Eigenvectors}$$

ω_i ***Natural Frequencies***

$\{\overline{D}\}_i$ ***Normal Modes***

Natural Frequencies

- [K] and [M] *n x n* then there are ***n*** eigenvalues and ***n*** eigenvectors
- [K] and [M] positive definite then eigenvalues are all positive
- $M_{ii} = 0$ infinite eigenvalue
- $M_{ii} < 0$ negative eigenvalue - imaginary frequency
- Use condensation to remove i^{th} equation if $M_{ii} = 0$

Rayleigh Quotient

$$\lambda = \frac{\{\overline{\mathbf{D}}\}^T [\mathbf{K}] \{\overline{\mathbf{D}}\}}{\{\overline{\mathbf{D}}\}^T [\mathbf{M}] \{\overline{\mathbf{D}}\}}$$

$[\mathbf{K}]$ ***symmetric***

$[\mathbf{M}]$ ***positive definite***

$\{\overline{\mathbf{D}}\}$ ***approximation to i^{th} eigenvector***

λ ***approximation to i^{th} eigenvalue***

Rayleigh Quotient

$$\lambda_{min} \le \frac{\{v\}^T [K]\{v\}}{\{v\}^T [M]\{v\}} \le \lambda_{max}$$

λ_{min} **smallest eigenvalue**

λ_{max} **largest eigenvalue**

Modal Methods

Modal Methods

When [K], [C], [M] are known and time independent the problem is linear.

$$[M]\{\ddot{D}\}+[C]\{\dot{D}\}+[K]\{D\}=\{F^{ext}\}$$

$\{\dot{D}(0)\},\{D(0)\}$ given as initial conditions

Modal Methods

Assume orthogonal damping, such as Rayleigh Damping. Modes can be uncoupled:

$$\{\bar{D}\}_i^T [M] \{\bar{D}\}_j = 0$$

$$\{\bar{D}\}_i^T [K] \{\bar{D}\}_j = 0$$

$$\{\bar{D}\}_i^T [C] \{\bar{D}\}_j = 0$$

$$i \neq j$$

$$\{\overline{D}\}_i^T [M] \{\overline{D}\}_i = 1$$

$$\{\overline{D}\}_i^T [K] \{\overline{D}\}_i = \omega_i^2$$

$$\{\overline{D}\}_i^T [C] \{\overline{D}\}_i = 2\xi_i \omega_i$$

Eigenvectors are linearly independent

$$[\phi] = \textit{matrix of eigenvectors (mode shapes)}$$

$$\{\mathbf{D}\} = [\phi]\{\mathbf{Z}\}$$

$$\{\mathbf{Z}\} \quad \textit{modal amplitudes}$$

Substitute into:

$$[M]\{\ddot{D}\}+[C]\{\dot{D}\}+[K]\{D\}=\{F^{ext}\}$$

$$\{\dot{D}(0)\},\{D(0)\} \quad given$$

$$[M][\phi]\{\ddot{Z}\}+[C][\phi]\{\dot{Z}\}+[K][\phi]\{Z\}=\{R^{ext}\}$$

$$[\phi]\{\dot{Z}(0)\}=\{\dot{D}(0)\}$$

$$[\phi]\{Z(0)\}=\{D(0)\} \quad \text{given}$$

Mode Displacement Method

$$[\mathbf{I}]\{\ddot{\mathbf{Z}}\}+[\xi]\{\dot{\mathbf{Z}}\}+[\omega^2]\{\mathbf{Z}\}=\{\mathbf{p}\}$$

Mode Displacement Method

Pre-multiply by $[\phi]^T$

$$[\phi]^T [M][\phi]\{\ddot{Z}\} + [\phi]^T [C][\phi]\{\dot{Z}\} + [\phi]^T [K][\phi]\{Z\} = [\phi]^T \{R^{ext}\}$$

$$[\phi]\{\dot{Z}(0)\} = \{\dot{D}(0)\}$$

$$[\phi]\{Z(0)\} = \{D(0)\} \quad \text{given}$$

Mode Displacement Method

$$[\phi]^T [M][\phi] = [I]$$

$$[\phi]^T [C][\phi] = [\xi]$$

$$[\phi]^T [K][\phi] = [\omega^2]$$

Mode Displacement Method

$$[\mathbf{I}]\{\ddot{\mathbf{z}}\}+[\xi]\{\dot{\mathbf{z}}\}+[\omega^2]\{\mathbf{z}\}=\{\mathbf{p}\}$$

Modes Uncouple:

$$[\mathbf{I}]\{\ddot{Z}\}+[\xi]\{\dot{Z}\}+[\omega^2]\{Z\}=\{Z\}$$

$$\ddot{Z}_i + 2\xi_i\omega_i\dot{Z}_i + \omega_i^2 Z = p_i \qquad i = 1, n$$

$$[\phi]\{\dot{Z}(0)\} = \{\dot{D}(0)\}$$

$$[\phi]\{Z(0)\} = \{D(0)\}$$

$$[\phi]^T [M][\phi]\{\dot{Z}(0)\} = [\phi]^T [M]\{\dot{D}(0)\}$$

$$[I]\{\dot{Z}(0)\} = [\phi]^T [M]\{\dot{D}(0)\}$$

$$\{\dot{Z}(0)\} = [\phi]^T [M]\{\dot{D}(0)\}$$

$$\{Z(0)\} = [\phi]^T [M]\{D(0)\}$$

Reduce size of problem:

$$m << n_{eq}$$

$$\{D\} = \sum_{i=1}^{m} \{\phi\}_i Z_i$$

Error Estimate:

$$e(t) = \frac{\left\| \{R^{ext}\} - [M]\{\ddot{D}\} - [C]\{\dot{D}\} - [K]\{D\} \right\|}{\left\| \{R^{ext}\} \right\|}$$

For an accurate analysis :

$$e(t) \leq 1\%$$

In many structural dynamics problems, more modes participate in the quasi-static response than in the dynamic response. For a small m value, the mode displacement method may have difficulty in predicting the quasi-static response.

Mode Acceleration method Method

$$[\mathbf{I}]\{\ddot{\mathbf{Z}}\}+[\xi]\{\dot{\mathbf{Z}}\}+[\omega^2]\{\mathbf{Z}\}=\{\mathbf{p}\}$$

Modal transformation only on inertial and damping terms

$$[M][\phi]\{\ddot{Z}\}+[C][\phi]\{\dot{Z}\}+[K]\{D\}=\{R^{ext}\}$$

$$[\phi]\{\dot{Z}(0)\}=\{\dot{D}(0)\}$$

$$[\phi]\{Z(0)\}=\{D(0)\} \quad \text{given}$$

$$[K]\{D\} = \{F^{ext}\} - [M][\phi]\{\ddot{Z}\} - [C][\phi]\{\dot{Z}\}$$

$$\{D\} = [K]^{-1}\{F^{ext}\} - [K]^{-1}\left([M][\phi]\{\ddot{Z}\} - [C][\phi]\{\dot{Z}\}\right)$$

$$[\phi]^T [M][\phi] = [I]$$

$$[M][\phi] = [\phi]^{-T}$$

$$\{D\} = [K]^{-1}\{F^{ext}\} - [K]^{-1}\left([\phi]^{-T}\{\ddot{Z}\} - [C][\phi]\{\dot{Z}\}\right)$$

$$[\phi]^T [K][\phi] = [\omega^2]$$

$$[\phi]^T [K][\phi][\omega^2]^{-1} = [I]$$

$$[K][\phi][\omega^2]^{-1} = [\phi]^{-T}[I] = [\phi]^{-T}$$

$$[\phi][\omega^2]^{-1} = [K]^{-1}[\phi]^{-T}$$

$$\{D\} = [K]^{-1}\{F^{ext}\} - [K]^{-1}[\phi]^{-T}\{\ddot{Z}\} + [K]^{-1}[C][\phi]\{\dot{Z}\}$$

$$\{D\} = [K]^{-1}\{F^{ext}\} - [\phi][\omega^2]^{-1}\{\ddot{Z}\} + [K]^{-1}[C][\phi]\{\dot{Z}\}$$

$$\{D\} = [K]^{-1}\{F^{ext}\} - [K]^{-1}[\phi]^{-T}\{\ddot{Z}\} + [K]^{-1}[C][\phi]\{\dot{Z}\}$$

$$\{D\} = [K]^{-1}\{F^{ext}\} - [\phi][\omega^2]^{-1}\{\ddot{Z}\} + [K]^{-1}[\phi]^{-T}[\phi]^{T}[C][\phi]\{\dot{Z}\}$$

$$\{D\} = [K]^{-1}\{F^{ext}\} - [\phi][\omega^2]^{-1}\{\ddot{Z}\} + [\phi][\omega^2]^{-1}[\xi]\{\dot{Z}\}$$

$$\{D\} = [K]^{-1}\{F^{ext}\} - [\phi][\omega^2]^{-1}\left(\{\ddot{Z}\} + [\xi]\{\dot{Z}\}\right)$$

$$\{D\} = [K]^{-1}\{F^{ext}\} - \sum_{i=1}^{m}\{\phi\}_i\left(\frac{1}{\omega_i^2}\ddot{Z}_i + \frac{2\xi_i}{\omega_i}\dot{Z}_i\right)$$

First term on RHS represents quasi-static response, the second term represents corrections for inertia and viscous effects.

Solve for Z terms as before:

$$\ddot{Z}_i + 2\xi_i\omega_i\dot{Z}_i + \omega_i^2 Z = p_i \qquad i = 1, n$$

Mass condensation

Mass Condensation

- Reduces number of d-o-f.
- Reduces expense of computing eigenvalues.
- Detrimental to accuracy.
- Not used with optimal lumping.

Guyan Reduction

$$\left(\left[\begin{array}{c|c} \mathbf{K}_{mm} & \mathbf{K}_{ms} \\ \hline \mathbf{K}_{ms}^{T} & \mathbf{K}_{ss} \end{array} \right] - \lambda \left[\begin{array}{c|c} \mathbf{M}_{mm} & \mathbf{M}_{ms} \\ \hline \mathbf{M}_{ms}^{T} & \mathbf{M}_{ss} \end{array} \right] \right) \left\{ \begin{array}{c} \overline{\mathbf{D}}_{m} \\ \hline \overline{\mathbf{D}}_{s} \end{array} \right\} = \left\{ \begin{array}{c} \mathbf{0} \\ \hline \mathbf{0} \end{array} \right\}$$

m - master degree of freedom

s - slave degree of freedom

Guyan Reduction

m - master degree of freedom
s - slave degree of freedom

Basic Assumption:
For lowest frequency modes the inertial forces on slave d-o-f are less important than elastic forces transmitted by master d-o-f. Thus we ignore all mass except M_{mm}

Guyan Reduction

$$\left(\left[\begin{array}{c|c} \mathbf{K}_{mm} & \mathbf{K}_{ms} \\ \hline \mathbf{K}_{ms}^{T} & \mathbf{K}_{ss} \end{array} \right] - \lambda \left[\begin{array}{c|c} \mathbf{M}_{mm} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} \end{array} \right] \right) \left\{ \begin{array}{c} \overline{\mathbf{D}}_{m} \\ \hline \overline{\mathbf{D}}_{s} \end{array} \right\} = \left\{ \begin{array}{c} \mathbf{0} \\ - \\ \mathbf{0} \end{array} \right\}$$

$$\left\{ \overline{\mathbf{D}}_{s} \right\} = -\left[\mathbf{K}_{ss} \right]^{-1} \left[\mathbf{K}_{ms} \right]^{T} \left\{ \overline{\mathbf{D}}_{m} \right\}$$

Guyan Reduction

$$\begin{Bmatrix} \overline{\mathbf{D}}_m \\ \hline \overline{\mathbf{D}}_s \end{Bmatrix} = [\mathbf{T}]\{\overline{\mathbf{D}}_m\}$$

$$[\mathbf{T}] = \begin{bmatrix} \mathbf{I} \\ \hline -\mathbf{K}_{ss}^{-1}\,\mathbf{K}_{ms}^{T} \end{bmatrix}$$

Guyan Reduction

$$\left([K_r] - \lambda [M_r] \right) \{\bar{D}_m\} = \{0\}$$

$$[K_r] = [T]^T [K][T]$$

$$[M_r] = [T]^T [M][T]$$

Both $[K_r]$ and $[M_r]$ are generally full.
$[M_r]$ contains both mass and stiffness terms

Guyan Reduction

$$[C_r] = [T]^T [C][T]$$

$$\{F_r^{ext}\} = [T]^T \{F^{ext}\}$$

$$[M_r]\{\ddot{D}_m\} + [C_r]\{\dot{D}_m\} + [K_r]\{D_m\} = \{F_r^{ext}\}$$

Compute Slaves

$$\{\overline{D}_s\}_i = -\left[K_{ss} - \lambda_i M_{ss}\right]^{-1}\left[K_{ms}^T - \lambda_i M_{ms}^T\right]\{\overline{D}_m\}_i$$

Choosing Master D-O-F

- Choose d-o-f where inertia is most important
- These are characterized by large mass to stiffness ratios.
- Each d-o-f that has a time varying applied load should be chosen.
- Master d-o-f should not be clustered.
- Process can be automated

Process for Choosing Master D-O-F

- Scan diagonal coefficients of [K] and [M].
- Choose d-o-f for which K_{ii}/M_{ii} is largest.
- This becomes first slave.
- Condense [K] and [M] by one order.
- Repeat process using condensed matrices till a user specified number of d-o-f remain.
- These are Master d-o-f chosen in near optimal way.

Number of Master D-O-F

- Choose cut-off frequency ω_c
- Take this to be about three times the highest frequency of interest.
- Terminate selection of master d-o-f when $K_{ii}/M_{ii} < \omega_c^2$
- Can combine manual and automatic selection (i.e. Choose each d-o-f that has a time varying applied load and then automatically select others.)
- Number of Master d-o-f may be 10% - 20% of total d-o-f.